Int. J. Solids Structures Vol. 20, No. 3, pp. 267 276, 1984
Printed in Great Britain.

0020-7683/84 $3.00 + .00

ON SINGULARITIES DUE TO A CONCENTRATED PRESSURE LOADING OF A CYLINDRICAL CAVITY†

R. PARNES
Department of Solid Mechanics, Materials and Structures, Faculty of Engineering, Tel-Aviv University, 69978 Ramat-Aviv, Israel

(*Received* 27 *December* 1982)

Abstract—A concentrated pressure loading applied to the surface of a cylindrical cavity within an elastic medium yields stress and displacement fields which are expressed by means of infinite integral representations. The resulting singularities in the stress and displacement components in the vicinity of the applied load are derived from these integrals and their evaluation leads to simple analytic expressions. The logarithmic and inverse power singularities obtained as well as jumps in the displacements are observed to coincide with those of the Flamant problem, as a limiting case.

1. INTRODUCTION

The problem of an elastic medium containing a cylindrical bore of radius a on whose surface a concentrated pressure is applied in the form of a circular line load was considered in[1]. Expressions for the stress and displacement fields were presented as integral representations over an infinite range $0 \leq \alpha < \infty$, where α is a parameter representing the ratio of a to a wave length. The integrals were evaluated and numerical results for the non-vanishing stress and displacement components along the radial axis and the bore surface were presented.

It was observed, however, that near the point of application of the applied concentrated loads, the convergence of the integrals was less rapid and that singularities occurred, as expected, in this zone. Accurate results were nevertheless given for points away from the point of loading.

In the present investigation, the singularities existing near the point of load application are determined, and their evaluation leads to simple analytic expressions. The results permit establishing some more general conclusions concerning the behaviour of the medium at points in the neighborhood of the applied traction. In addition, the results are applicable to the determination of stresses in the context of hydraulic fracture problems.

2. EVALUATION OF THE SINGULARITIES.

We consider an infinite isotropic elastic medium (whose shear modulus is μ and with a Poisson ratio ν) containing a cylindrical bore of radius $r = a$. The medium is referred to a non-dimensional cylindrical coordinate system ($\rho = r/a, \theta, \zeta = z/a$). An axi-symmetrical radial line load of intensity P is applied along a circle of the bore at $\zeta = 0$ (Fig. 1). The resulting boundary conditions are then

$$\sigma_{rr} = -\frac{P}{a}\delta(\zeta),\ \ \sigma_{r\theta} = \sigma_{rz} = 0;\ \ \rho = 1 \tag{1}$$

where $\delta(\zeta)$ is the Dirac-delta function.

The axi-symmetric displacements are denoted by

$$\tilde{u}(\rho, \zeta) = w\tilde{k}_r + u\tilde{k}_z \tag{2}$$

and σ_{ij} $(i, j = r, \theta, z)$ denote the stress components.

†This work was initiated while the author was on leave at Laboratoire de Mécanique des Solides, Ecole Polytechnique, Palaiseau, France.

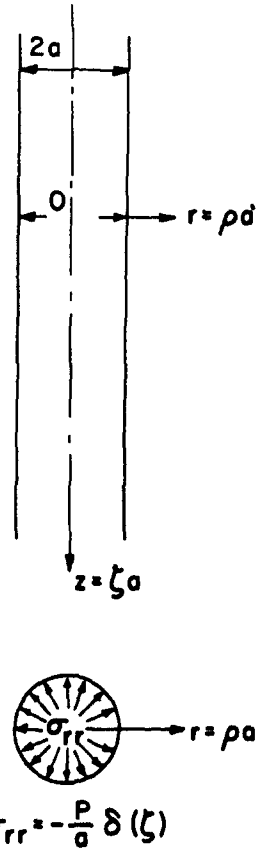

Fig. 1. Geometry of problem.

The non-dimensional displacement and stress fields are then given by

$$\left.\begin{aligned}\frac{\mu u_i}{P}&=\frac{1}{\pi}\int_0^\infty Q_i(\alpha,\rho)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\\ \frac{a\sigma_{ij}}{P}&=\frac{1}{\pi}\int_0^\infty Q_{ij}(\alpha,\rho)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\end{aligned}\right\}\qquad(3)$$

where, in the above, u and σ_{rz} are associated with $\sin\alpha\zeta$ and the non-vanishing remaining quantities, $(w, \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz})$ with $\cos\alpha\zeta$. The expressions for Q_i and Q_{ij} are presented in[1] and are repeated here for convenience and consistency of nomenclature:

$$\left.\begin{aligned}
Q_u&=\frac{1}{2D}\{\rho\alpha K_1K_1(\alpha\rho)-2(1-\nu)K_1K_0(\alpha\rho)+\alpha K_0K_0(\alpha\rho)\}\text{(a)}\\
Q_w&=\frac{1}{2D}\{\rho\alpha K_0K_0(\alpha\rho)+2(1-\nu)K_1K_1(\alpha\rho)-\alpha K_0K_1(\alpha\rho)\}\text{(b)}\\
Q_{rr}&=\frac{1}{D}\left\{\alpha^2K_0K_0(\alpha\rho)+\frac{\alpha}{\rho}K_0K_1(\alpha\rho)-\alpha K_1K_0(\alpha\rho)-[2(1-\nu)/\rho+\alpha^2\rho]K_1K_1(\alpha\rho)\right\}\text{(c)}\\
Q_{\theta\theta}&=\frac{1}{D}\left\{-\frac{\alpha}{\rho}K_0K_1(\alpha\rho)+\frac{2(1-\nu)}{\rho}K_1K_1(\alpha\rho)+(1-2\nu)\alpha K_1K_0(\alpha\rho)\right\}\\
Q_{zz}&=\frac{\alpha}{D}\{\alpha\rho K_1K_1(\alpha\rho)-2K_1K_0(\alpha\rho)-\alpha K_1K_1(\alpha\rho)\}\text{(e)}\\
Q_{rz}&=\frac{\alpha^2}{D}\{K_0K_1(\alpha\rho)-\rho K_1K_0(\alpha\rho)\}\text{(f)}
\end{aligned}\right\}\qquad(4)\dagger$$

†For simplification, the following notation is used here and below: $K_n \equiv K_n(\alpha)$.

where

$$D = [\alpha^2 + 2(1-\nu)]K_1^2 - \alpha^2 K_0^2. \tag{5}$$

In the above, K_n represent the modified Bessel functions of order n.

The infinite integrals appearing in eqn (3) (which we denote in general by S_i or S_{ij}) can be evaluated by choosing prescribed finite values of α, $\alpha = \alpha_0$, thus separating the infinite range into two subranges $0 \le \alpha \le \alpha_0$ and $\alpha_0 \le \alpha < \infty$.

The resulting integrals in the two ranges, denoted by $S^{(1)}$ and $S^{(2)}$ respectively, such that

$$S_i = S_i^{(1)} + S_i^{(2)} \text{ or } S_{ij} = S_{ij}^{(1)} + S_{ij}^{(2)}, \tag{6}$$

are then integrated in turn: the $S^{(1)}$ integrals are integrated numerically and asymptotic approximations to the $S^{(2)}$ integrals are obtained as described below. The integrands of $S^{(1)}$ are found to be bounded and well-behaved in all cases for $\rho \ge 1$ and $\zeta \ge 0$ and no particular problems arise in the numerical integration procedure ever the range $0 < \alpha \le \alpha_0$.

If α_0 is chosen sufficiently large ($\alpha_0 \gg 1$), then the K_n functions appearing in eqns (4) may be represented, for $\alpha > \alpha_0$, by their asymptotic expansions[2]

$$K_n(x) = \sqrt{\pi/2x}\, e^{-x}[1 + c_n/x + d_n/x^2 + \ldots] \tag{7a}$$

where

$$c_n = \frac{4n^2 - 1^2}{1!8},\ d_n = \frac{(4n^2-1^2)(4n^2-3^2)}{2!8^2}. \tag{7b}$$

Upon substituting in eqn (4) and noting that the numerators appear as products of $K_n(\alpha)\,K_m(\alpha\rho)$, the integrals $S^{(2)}$ for all the desired quantities, are seen to have the general form

$$S^{(2)} = \frac{1}{\pi}\int_{\alpha_0}^{\infty} e^{-(\rho-1)\alpha} g(\alpha,\rho) \begin{Bmatrix} \cos\alpha\zeta \\ \sin\alpha\zeta \end{Bmatrix} d\alpha \tag{8}$$

where $g(\alpha, \rho)$ is a function expressed as a Laurent series in powers of α.

Qualitatively, we immediately observe that the decay in the integrands will be rather fast provided ρ is sufficiently greater than unity. However if ρ approaches unity, it becomes obvious that the decay will be rather slow and would depend on the particular Laurent series representation of $g(\alpha, \rho)$. Thus, as may be seen from eqn (8), problems are expected to arise in the convergence of the $S^{(2)}$ integrals as $\rho \to 1$. Indeed, in evaluating the integrals $S^{(2)}$ presented in [1], convergence was found to be sufficiently rapid for $\rho \gtrsim 1.1$ and $\zeta \gtrsim 0.1$. For such cases the $S^{(2)}$ integrals were found to represent a small contribution to the total integrals S. However, as $\rho \to 1$ and $\zeta \to 0$, the $S^{(2)}$ integrals become increasingly large and, in fact, these integrals will be shown to contain the singularities which occur in the region near the point of the applied loads. Thus, the slow convergence of the $S^{(2)}$ integrals as $\rho \to 1$ or $\zeta \to 0$ is, in a sense, a reflection of the existence of the singularities. We therefore seek to extract the singularities from the integral expressions for $S^{(2)}$, and proceed as follows.

Using eqns (7) for $\alpha \ge \alpha_0$ and retaining the first three terms of the representation, the expansions of the Q integrands which appear in eqn (4) can, after some manipulation, (details of which are given in the Appendix) be expressed in the following form:

For the displacements:

$$Q_i = \rho^{-1/2} e^{-(\rho-1)\alpha}[\beta_i^{(0)} + \beta_i^{(1)}/\alpha];\quad \alpha_0 \le \alpha$$

For the stresses:

$$Q_{ij} = \rho^{-1/2} e^{-(\rho-1)\alpha}[\beta_{ij}^{*}\alpha + \beta_{ij}^{(0)} + \beta_{ij}^{(1)}/\alpha];\ \alpha_0 \le \alpha \tag{9}$$

where

$$\beta_i = \beta_i(\rho, \nu), \ \beta_{ij} = \beta_{ij}(\rho, \nu). \tag{10}$$

Values of the β coefficients are given in the Appendix.

Now, from eqn (6), we may rewrite eqns (3) as

$$\frac{\mu u_i}{P} = \frac{1}{\pi}\left[\int_0^{\alpha_0} Q_i(\alpha, \rho)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha + \int_{\alpha_0}^{\infty} Q_i(\alpha, \rho)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\right] \tag{11a}$$

and

$$\frac{a\sigma_{ij}}{P} = \frac{1}{\pi}\left[\int_0^{\alpha_0} Q_{ij}(\alpha, \rho)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}s + \int_{\alpha_0}^{\infty} Q_{ij}(\alpha, \rho)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\right]. \tag{11b}$$

It is noted that the expressions for Q_i and Q_{ij}, given by eqns (9) are valid only in the right hand integrals (α_0 to ∞) of eqns (11).

Upon examining eqn (9), it is clear that the main contribution to the $S^{(2)}$ integrals are due to the β^* and $\beta^{(0)}$ terms. We therefore define, for the displacements:

$$R_i \equiv \rho^{-1/2}\,\mathrm{e}^{-(\rho-1)\alpha}\beta_i^{(0)}, \tag{12a}$$

and for the stresses

$$R_{ij} \equiv \rho^{-1/2}\,\mathrm{e}^{-(\rho-1)\alpha}[\beta_{ij}^*\alpha + \beta_{ij}^{(0)}]. \tag{12b}$$

The functions R_i and R_{ij} are observed to be well behaved and bounded for all $0 < \alpha \leq \alpha_0$. We now rewrite eqns (11) as follows:

$$\frac{\mu u_i}{P} = \frac{1}{\pi}\left[\int_0^{\alpha_0}(Q_i - R_i)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha + \int_{\alpha_0}^{\infty}(Q_i - R_i)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha + \int_0^{\infty} R_i\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\right] \tag{13a}$$

$$\frac{a\sigma_{ij}}{P} = \frac{1}{\pi}\left[\int_0^{\alpha_0}(Q_{ij} - R_{ij})\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha + \int_{\alpha_0}^{\infty}(Q_{ij} - R_{ij})\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha + \int_0^{\infty} R_{ij}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\right]. \tag{13b}$$

From eqns (9) and (12) we then obtain

$$\begin{aligned}\frac{\mu u_i}{P} = \frac{1}{\pi}\Bigg[&\int_0^{\alpha_0}(Q_i - R_i)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha + \rho^{-1/2}\beta_i^{(1)}\int_{\alpha_0}^{\infty}\frac{\mathrm{e}^{-(\rho-1)\alpha}}{\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\\ &+\rho^{-1/2}\beta_i^{(0)}\int_0^{\infty}\mathrm{e}^{-(\rho-1)\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\Bigg]\end{aligned} \tag{14a}$$

$$\begin{aligned}\frac{a\sigma_{ij}}{P} = \frac{1}{\pi}\Bigg[&\int_0^{\alpha_0}(Q_{ij} - R_{ij})\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha + \rho^{-1/2}\beta_{ij}^{(1)}\int_{\alpha_0}^{\infty}\frac{\mathrm{e}^{-(\rho-1)\alpha}}{\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\\ &+\rho^{-1/2}\int_0^{\infty}[\beta_{ij}^*\alpha + \beta_{ij}^{(0)}]\,\mathrm{e}^{-(\rho-1)\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha\Bigg].\end{aligned} \tag{14b}$$

It is noted that the first right hand integrals appearing in eqns (14) are again well behaved and bounded and create no difficulties when a numerical integration procedure is performed over the finite range $0 \leq \alpha \leq \alpha_0$.

Thus, eqns (11) are finally replaced by

$$\frac{\mu u_i}{P}=\frac{1}{\pi}\left[\int_0^{\alpha_0}(Q_i-R_i)\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha+\left[\frac{\rho^{-1/2}\beta_i^{(1)}}{\pi}\right]I+\frac{\rho^{-1/2}\beta_i^{(0)}}{\pi}\Gamma_1\right] \tag{15a}$$

$$\frac{a\sigma_{ij}}{P}=\frac{1}{\pi}\int_0^{\alpha_0}(Q_{ij}-R_{ij})\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha+\left[\frac{\rho^{-1/2}\beta_{ij}^{(1)}}{\pi}\right]I+\frac{\rho^{-1/2}}{\pi}[\beta_{ij}^{*}\Gamma_1+\beta_{ij}^{(0)}\Gamma_2] \tag{15b}$$

where

$$I=\int_{\alpha_0}^{\infty}\frac{\mathrm{e}^{-(\rho-1)\alpha}}{\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha \tag{16}$$

and

$$\Gamma_1(\rho,\zeta)=\int_0^{\infty}\mathrm{e}^{-(\rho-1)\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha \tag{17a}$$

$$\Gamma_2(\rho,\zeta)=\int_0^{\infty}\alpha\,\mathrm{e}^{-(\rho-1)\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha. \tag{17b}$$

The integrals Γ_1 and Γ_2 are easily integrated[3]

$$\int_0^{\infty}\mathrm{e}^{-(\rho-1)\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha=\frac{1}{[(\rho-1)^2+\zeta^2]}\begin{Bmatrix}(\rho-1)\\ \zeta\end{Bmatrix}$$

$$\int_0^{\infty}\alpha\,\mathrm{e}^{-(\rho-1)\alpha}\begin{Bmatrix}\cos\alpha\zeta\\ \sin\alpha\zeta\end{Bmatrix}\mathrm{d}\alpha=\frac{1}{[(\rho-1)^2+\zeta^2]^2}\begin{Bmatrix}[(\rho-1)^2-\zeta^2]\\ 2(\rho-1)\zeta\end{Bmatrix}. \tag{18a, b}$$

For arbitrary values of ρ and ζ, the integrals I, in the range α_0 to ∞, in general, cannot be integrated analytically. However, along the radial line $\zeta=0$ and the boundary $\rho=1$, these integrals yield analytic expressions. We therefore investigate these two cases further.

(a) *Behaviour along* $\zeta=0$ *as* $\rho\to 1$

For this case, $u=\sigma_{rz}=0$ while the remaining components become

$$\mu w/P=\frac{1}{\pi}\left[\int_0^{\alpha_0}(Q_w-R_w)\,\mathrm{d}\alpha+\frac{\rho^{1/2}\beta_w^{(1)}}{\pi}\int_{\alpha_0}^{\infty}\frac{\mathrm{e}^{-(\rho-1)\alpha}}{\alpha}\mathrm{d}\alpha+\frac{\rho^{-1/2}\beta_w^{(0)}}{\pi(\rho-1)}\right] \tag{19a}$$

$$a\sigma_{ij}/P=\frac{1}{\pi}\left[\int_0^{\alpha_0}(Q_{ij}-R_{ij})\,\mathrm{d}\alpha+\frac{\rho^{-1/2}\beta_{ij}^{(1)}}{\pi}\int_{\alpha_0}^{\infty}\frac{\mathrm{e}^{-(\rho-1)\alpha}}{\alpha}\mathrm{d}\alpha+\frac{\rho^{-1/2}}{\pi}\right.$$
$$\left.\times[\beta_{ij}^{(0)}\Gamma_1+\beta_{ij}^{*}\Gamma_2]\right]. \tag{19b}$$

Using the definition of the exponential integral function[4],

$$E_1(\alpha_0)=\int_{\alpha_0}^{\infty}\frac{\mathrm{e}^{-x}}{x}\mathrm{d}x, \tag{20}$$

and noting that

$$E_1(a\alpha_0) = \int_{\alpha_0}^{\infty} \frac{e^{-ax}}{x}\, dx, \tag{21}$$

we rewrite eqns (19) as

$$\mu w/P = \frac{1}{\pi}\left[\int_0^{\alpha_0} (Q_w - R_w)\, d\alpha + \frac{\rho^{-1/2}\beta_w^{(1)}}{\pi} E_1[(\rho-1)\alpha_0] + \frac{\rho^{-1/2}\beta_w^{(0)}}{\pi(\rho-1)}\right. \tag{22a}$$

$$a\sigma_{ij}/P = \frac{1}{\pi}\left[\int_0^{\alpha_0} (Q_{ij} - R_{ij})\, d\alpha + \frac{\rho^{-1/2}\beta_{ij}^{(1)}}{\pi} E_1[(\rho-1)\alpha_0] + \frac{\rho^{-1/2}}{\pi}\right.$$
$$\times [\beta_{ij}^{(0)}\Gamma_1 + \beta_{ij}^{*}\Gamma_2]. \tag{22b}$$

Now, $E_1[(\rho-1)\alpha_0]$ can be represented by the series expansion[4]

$$E_1[(\rho-1)\alpha_0] = -\gamma - \ln[(\rho-1)\alpha_0] + (\rho-1)\alpha_0 - (\rho-1)^2\alpha_0^2/4 + \ldots \tag{23}$$

where $\gamma = 0.5772157\ldots$. Hence we observe that this term produces a logarithmic singularity as $\rho \to 1$. Examining the last term of eqn (22a) we note that since $\beta_w^{(0)} \sim \rho - 1$, no further singularity occurs in the displacement w. However for the stresses σ_{ij}, the last term appearing in eqn (22b) may produce strong singularities upon taking the $\lim_{\rho\to 1} \rho^{-1/2}[\beta_{ij}^{(0)}\Gamma_1 + \beta_{ij}^{*}\Gamma_2]$.

We denote the singularities for the various quantities by Σ_i or Σ_{ij}. Using the values of the coefficients of $\beta_i^{(n)}$ and $\beta_{ij}^{(n)}$, $n = 0, 1$, as given by eqns (A6) and taking the limit as $\rho \to 1$, we obtain.

$$\left.\begin{aligned}
\Sigma_w &= -\frac{P}{\pi\mu}(1-\nu)\ln(\rho-1) \to \infty \\
\Sigma_{rr} &= -\frac{2P}{\pi a}(\rho-1)^{-1} \to -\infty \\
\Sigma_{\theta\theta} &= -\frac{P}{\pi a}[(2+\nu-4\nu^2)\ln(\rho-1) + 2\nu(\rho-1)^{-1}] \to \begin{cases} -\infty, & \nu \neq 0 \\ +\infty, & \nu = 0 \end{cases} \\
\Sigma_{zz} &= -\frac{P}{\pi a}(1-2\nu)\ln(\rho-1) \to \infty.
\end{aligned}\right\} \tag{24}$$

(b) *Behaviour along boundary $\rho = 1$ as $\zeta \to 0$*

For this case, the prescribed boundary conditions ($\zeta > 0$) are $\sigma_{rr} = \sigma_{rz} = 0$, while $\sigma_{rr} = -P/a\,\delta(\zeta)$ is the prescribed singularity at $\zeta = 0$

With $\rho = 1$, eqns (15) and (16) yield

$$\mu u_i/P = \frac{1}{\pi}\int_0^{\alpha_0} (Q_i - R_i)\begin{Bmatrix}\cos\alpha\zeta \\ \sin\alpha\zeta\end{Bmatrix} d\alpha + \frac{\beta_i^{(1)}}{\pi}\int_{\alpha_0}^{\infty} \frac{1}{\alpha}\begin{Bmatrix}\cos\alpha\zeta \\ \sin\alpha\zeta\end{Bmatrix} d\alpha + \frac{\beta_i^{(0)}}{\pi}\Gamma_1 \tag{25a}$$

$$a\sigma_{ij}/P = \frac{1}{\pi}\int_0^{\alpha_0} (Q_{ij} - R_{ij})\begin{Bmatrix}\cos\alpha\zeta \\ \sin\alpha\zeta\end{Bmatrix} d\alpha + \frac{\beta_{ij}^{(1)}}{\pi}\int_{\alpha_0}^{\infty} \frac{1}{\alpha}\begin{Bmatrix}\cos\alpha\zeta \\ \sin\alpha\zeta\end{Bmatrix} d\alpha + \frac{\beta_{ij}^{(0)}}{\pi}\Gamma_2 \tag{25b}$$

since, from eqns (A7), $\beta_{ij}^{*} = 0$ at $\rho = 1$ for all i and j.

Using the definitions of the cosine- and sine integrals respectively [4]

$$Ci(\alpha_0) = -\int_{\alpha_0}^{\infty} \frac{\cos x}{x}\, dx, \quad si(\alpha_0) = -\int_{\alpha_0}^{\infty} \frac{\sin x}{x}\, dx \tag{26a, b}$$

and proceeding as before, we obtain for the displacements

$$\mu u/P = \frac{1}{\pi}\int_0^{\alpha_0} (Q_u - R_u)\sin\alpha\zeta \, d\alpha + \frac{(1-2\nu)}{2\pi} si(\alpha_0\zeta) \tag{27a}$$

$$\mu w/P = \frac{1}{\pi}\int_0^{\alpha_0} (Q_w - R_w)\cos\alpha\zeta \, d\alpha - \frac{(1-\nu)}{\pi} Ci(\alpha_0\zeta) \tag{27b}$$

where, in the above, the proper coefficients for $\beta_i^{(n)}|_{\rho=1}$ $(n = 0, 1)$, from eqn (A7), have been inserted.

For the non vanishing stresses, we obtain, (upon noting that $R_{ij}|_{\rho=1} = \beta_{ij}^{(0)}|_{\rho=1}$) from eqn (13b), and upon inserting the definition of eqn (26):

$$\sigma_{ij}a/P = \frac{1}{\pi}\int_0^{\alpha_0} (Q_{ij} - R_{ij})\cos\alpha\zeta \, d\alpha - \frac{\beta_{ij}^{(1)}}{\pi} Ci(\alpha_0\zeta) + \frac{\beta_{ij}^{(0)}(\rho = 1)}{\pi}\int_0^{\infty} \cos\alpha\zeta \, d\alpha. \tag{28}$$

We first observe that $si(\alpha_0\zeta)$ appearing in eqn (27a) is always bounded. Consequently no singularity appears in the longitudinal displacement u as $\zeta \to 0$. Using the series representation[4]

$$Ci(\alpha\zeta) = \gamma + \ln(\alpha_0\zeta) - (\alpha_0\zeta)^2/4 + (\alpha_0\zeta)^4/96 - \ldots \tag{29}$$

we observe that a logarithmic singularity occurs in the displacement w and in the stresses σ_{ij} (provided $\beta_{ij}^{(1)} \neq 0$) as $\zeta \to 0$.

The last term appearing in eqn (28) is recognized as the integral representation of the Dirac-delta function $\delta(\zeta)$ [5]; i.e.

$$\frac{1}{\pi}\int_0^{\infty} \cos\alpha\zeta \, d\alpha = \delta(\zeta). \tag{30}$$

Since $\delta(\zeta) = 0$ for $\zeta \neq 0$, we disregard this term for all $\zeta > 0$.

Using then, eqn (29) and the values of $\beta_{ij}^{(n)}(\rho = 1)$ given in eqn (A7) we obtain the following singularities:

$$\left.\begin{aligned}
\Sigma_u &= 0\\
\Sigma_w &= -\frac{P}{\pi\mu}(1-\nu)\ln\zeta \to \infty\\
\Sigma_{\theta\theta} &= -\frac{P}{\pi a}(2+\nu-4\nu^2)\ln\zeta \to \infty\\
\Sigma_{zz} &= -\frac{P}{\pi a}(1-2\nu)\ln\zeta \to \infty.
\end{aligned}\right\} \tag{31}$$

3. DISCUSSION

The singularities for the displacement and stress components as $\rho \to 1$ and $\zeta \to 0$ are given by eqns (24) and (31) respectively. In addition, at the point of loading ($\rho = 1$, $\zeta = 0$), Dirac-delta singularities of strength $\beta_{ij}^{(0)}$ exist for the non-vanishing stress components.

In the region of loading, it is observed that along the radial line $\zeta = 0$, the radial displacement has a logarithmic singularity while strong singularities, $0[(\rho - 1)^{-1}]$, exist for the stress components σ_{rr} and $\sigma_{\theta\theta}$ (when $\nu > 0$). The singularities are observed to satisfy the stress–strain relations in the neighborhood of the applied load. It is instructive to verify these relations. For example, in examining the relation $\epsilon_{\theta\theta} = 1/E[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})]$ we observe that for the strain $\epsilon_{\theta\theta}$, in the case $\nu > 0$, the effect of σ_{rr} cancels with the dominant singularity in

$\sigma_{\theta\theta}$; thus this strain component results physically due to the predominant behavior of $\sigma_{\theta\theta}$ and σ_{zz}. It is of interest to observe, too, that for $\nu = 0$ the singularity in the stress $\sigma_{\theta\theta}$ is weakened, becoming logarithmic.

Along the boundary $\rho = 1$, the radial displacement and stress components all have logarithmic singularities as $\zeta \to 0$, while no singularity occurs in the longitudinal displacement u. However, from eqn (27a) we note, upon letting $\zeta \to 0$, that

$$\lim_{\zeta \to 0} \frac{\mu u}{P}\bigg|_{\rho=1} = -\frac{1-2\nu}{4} \tag{32}$$

since $si(x \to 0) = -\pi/2$ [4]. This result appears to be in contradiction with eqns (3) and (4a) which lead to $u(\zeta = 0) = 0$, and therefore implies a jump in u at the point of loading. To investigate this phenomenon, we examine the expression for $\partial u/\partial\zeta$. From eqns (3)

$$\frac{\mu}{P}\partial u/\partial\zeta\bigg|_{\rho=1} = \frac{1}{\pi}\int_0^\infty \alpha\, Q_u(\alpha, 1)\cos\alpha\zeta\, \mathrm{d}\alpha. \tag{33}$$

Following the same method as given above and in the Appendix, we obtain

$$\frac{\mu}{P}\frac{\partial u}{\partial\zeta}\bigg|_{\rho=1} = \frac{1}{\pi}\left[\int_0^{\alpha_0}(\alpha Q_u - \beta_u^{(1)})\bigg|_{\rho=1}\cos\alpha\zeta\,\mathrm{d}\alpha - \frac{\beta_u^{(2)}}{2\pi}\bigg|_{\rho=1} Ci(\alpha_0\zeta) + \frac{\beta_u^{(1)}}{\pi}\bigg|_{\rho=1}\right.$$
$$\times \int_0^\infty \cos\alpha\zeta\,\mathrm{d}\alpha \tag{34a}$$

where

$$\beta_u^{(2)}\bigg|_{\rho=1} = 5/4 - \nu(11/2\text{–}4\nu). \tag{34b}$$

As before, the Ci term gives a logarithmic singularity. Again we observe, from eqn (30), that the last term represents the Dirac-delta function. Thus, we have

$$\frac{\partial u}{\partial\zeta}\bigg|_{\substack{\rho=1\\ \zeta=0}} \doteq \frac{P\beta_u^{(1)}}{\mu}\bigg|_{\rho=1}\delta(\zeta) \tag{35}$$

Using the fundamental property,

$$\int_{-\zeta}^{\zeta}\delta(x)\,\mathrm{d}x = 1, \zeta > 0 \tag{36}$$

we obtain

$$\int_{-\zeta}^{\zeta}\mathrm{d}u = \int_{-\zeta}^{\zeta}\frac{\partial u}{\partial\zeta}\,\mathrm{d}\zeta = \frac{P}{\mu}\beta_u^{(1)}\bigg|_{\rho=1} \tag{37a}$$

i.e.

$$u(\zeta) - u(-\zeta) = -\frac{P}{2\mu}(1-2\nu). \tag{37b}$$

Since $u(\zeta) = -u(-\zeta)$, we recover eqn (32). Thus, although no singularity in u occurs at the point ($\rho = 1, \zeta = 0$), a jump in the displacement does indeed exist and is due to the Dirac-delta in the derivative. Such jumps in displacements are known to occur under concentrated loads (see, e.g. [6]).

Finally, it is of interest to note that the classical Flamant problem[7] represents the degenerate (plane stress) counterpart of the present problem as $a \to \infty$. Upon making the appropriate adjustments in the material constants, the logarithmic and inverse power singularities as well as the jump in the longitudinal displacement of the two solutions are seen to coincide.

REFERENCES

1. R. Parnes, Applied tractions on the surface of an infinite cylindrical bore. *Int. J. Solids Structures* **19**, 165–172 (1983).
2. N. W. McLachlan, *Bessel Functions for Engineers*. Oxford (1955).
3. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*. Academic Press, New York (1965).
4. M. Abramowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions*. Dover, New York (1965).
5. I. N. Sneddon, *Fourier Transforms*. McGraw-Hill, New York (1951).
6. R. Parnes and E. R. Johnson, Axi-symmetric loads on an elastic half-space containing an inclusion. *Int. J. Engng Sci.* **11**, 717-724 (1973).
7. S. Timoshenko and J. N. Goodier, *Theory of Elasticity* (3rd Edn). McGraw-Hill, New York (1970).

APPENDIX

Asymptotic expansion of the Q integrands. Results for β coefficients

Using the asymptotic expansions of the K_n functions (up to the third term), given by eqn (7), the products of the various K_n functions appearing in eqns (4) and (5), upon collecting powers of α, are

$$\left.\begin{aligned} K_nK_n(\alpha\rho) &= \frac{A}{\alpha}[1 + a_{n1}/\alpha + a_{n2}/\alpha^2 + a_{n3}/\alpha^3],\ n = 0, 1 \\ K_0K_1(\alpha\rho) &= \frac{A}{\alpha}[1 + b_{01}/\alpha + b_{02}/\alpha^2 + b_{03}/\alpha^3] \\ K_1K_0(\alpha\rho) &= \frac{A}{\alpha}[1 + b_{11}/\alpha + b_{12}/\alpha^2 + b_{13}/\alpha^3] \end{aligned}\right\} \tag{A1}$$

where

$$\left.\begin{aligned} a_{01} &= -\frac{1}{8}(1 + 1/\rho),\ a_{02} = \frac{1}{128}(9 + 2/\rho + 9/\rho^2),\ a_{03} = -\frac{1}{1024}(75 + 9/\rho + 9/\rho^2 + 75/\rho^3) \\ a_{11} &= \frac{3}{8}(1 + 1/\rho),\ a_{12} = \frac{1}{128}(-15 + 18/\rho - 15/\rho^2),\ a_{13} = \frac{1}{1024}(105 - 45/\rho - 45/\rho^2 + 105/\rho^3) \\ b_{01} &= \frac{1}{8}(-1 + 3/\rho),\ b_{02} = \frac{1}{128}(9 - 6/\rho - 15/\rho^2),\ b_{03} = \frac{1}{1024}(-75 + 27/\rho + 15/\rho^2 + 105/\rho^3) \\ b_{11} &= \frac{1}{8}(3 - 1/\rho),\ b_{12} = \frac{1}{128}(-15 - 6/\rho + 9/\rho^2),\ b_{13} = \frac{1}{1024}(105 + 15/\rho + 27/\rho^2 - 75/\rho^3) \end{aligned}\right\} \tag{A2}$$

and

$$A = (\pi/2)\rho^{-1/2}\,e^{-(\rho+1)\alpha}. \tag{A3}$$

Substituting in eqn (5), $D = (\pi/2)\,e^{-2\alpha}\{1 + (7/4 - 2\nu)/\alpha + \ldots\} = (\pi/2)\,e^{-2\alpha}\{1 - (7/4 - 2\nu)/\alpha\ldots\}^{-1}$ (A4)

for $\alpha \gg 1$.

Substituting eqns (A1) and the last of (A4) in eqns (4) and collecting terms of the same order, the integrands Q_i and Q_{ij} become, for $1 \ll \alpha$:

$$\begin{aligned} Q_i &= \rho^{-1/2}\,e^{-(\rho-1)\alpha}\{\beta_i^{(0)} + \beta_i^{(1)}/\alpha\} \\ Q_{ij} &= \rho^{-1/2}\,e^{-(\rho-1)\alpha}\{\beta_{ij}^*\alpha + \beta_{ij}^{(0)} + \beta_{ij}^{(1)}/\alpha\} \end{aligned} \tag{A5}$$

where

$$\beta_u^{(0)} = 1/2(\rho - 1),\ \beta_u^{(1)} = \frac{1}{16}(2 - 11\rho + 1/\rho) + \nu\rho \tag{A6a}$$

$$\beta_w^{(0)} = 1/2(\rho - 1),\ \beta_w^{(1)} = \frac{1}{16}(30 - 11\rho - 3/\rho) - \nu(2 - \rho) \tag{A6b}$$

$$\left.\begin{aligned} &\beta_{rr}^* = 1 - \rho,\ \beta_{rr}^{(0)} = \frac{1}{8}(-26 + 7/\rho + 11\rho) - 2\nu(\rho - 1), \\ &\beta_{rr}^{(1)} = \frac{1}{128}(279 + 99\rho - 435/\rho + 57/\rho^2) - \frac{3\nu}{4}(4 + \rho - 5/\rho) \end{aligned}\right\} \tag{A6c}$$

$$\left.\begin{aligned} &\beta_{\theta\theta}^* = 0,\ \beta_{\theta\theta}^{(0)} = 1 - 1/\rho - 2\nu \\ &\beta_{\theta\theta}^{(1)} = \frac{1}{8}(-11 + 30/\rho - 3/\rho^2) + \frac{\nu}{4}(19 - 15/\rho - 16\nu) \end{aligned}\right\} \tag{A6d}$$

$$\left.\begin{aligned} \beta^*_{zz} = \rho - 1,\ \beta^{(0)}_{zz} &= \frac{1}{8}(2 - 11\rho + 1/\rho) - 2\nu(1-\rho) \\ \beta^{(1)}_{zz} &= \frac{1}{128}(249 - 99\rho - 13/\rho - 9/\rho^2) - \frac{\nu}{4}(12 - 3\rho - 1/\rho) \end{aligned}\right\} \quad \text{(A6e)}$$

$$\left.\begin{aligned} \beta^*_{rz} = -(\rho - 1),\ \beta^{(0)}_{rz} &= \frac{1}{8}(-14 + 11\rho + 3/\rho) + 2\nu(1-\rho) \\ \beta^{(1)}_{rz} &= \frac{1}{128}(15 + 99\rho - 99/\rho - 15/\rho^2) - 3\nu/4(\rho - 1/\rho) \end{aligned}\right\} \quad \text{(A6f)}$$

For the case $\rho = 1$, the coefficients become;

$$\beta^*_{ij} = 0, \text{all } i, j = r, \theta, z \quad \text{(A7a)}$$

$$\beta^{(0)}_{rr} = -1,\ \beta^{(0)}_{\theta\theta} = -2\nu,\ \beta^{(0)}_{zz} = -1,\ \beta^{(0)}_{u} = \beta^{(0)}_{w} = \beta^{(0)}_{rz} = 0 \quad \text{(A7b)}$$

$$\left.\begin{aligned} &\beta^{(1)}_{\theta\theta} = 2 + \nu - 4\nu^2,\ \beta^{(1)}_{zz} = 1 - 2\nu,\ \beta^{(1)}_{u} = -\frac{1}{2}(1 - 2\nu),\ \beta^{(1)}_{w} = 1 - \nu \\ &\beta^{(1)}_{rr} = \beta^{(1)}_{rz} = 0 \end{aligned}\right\} \quad \text{(A7c)}$$